

A NOTE ON THE VARIETY OF SECANT LOCI

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ABSTRACT. We determine non-hyper elliptic curves C with $g(C) \geq 9$, such that for some very ample line bundle L on them and for some integers d and r with $0 < 2r < d \leq h^0(L) + r - 4$, the dimension of the Secant Loci, $\dim V_d^{d-r}(L)$, attains one less than its maximum value. Then we proceed to prove that for positive integers γ, d with some circumstances on γ, d and $h^0(L)$, if one had $\dim V_d^{d-1}(L) = d - 1 - \gamma$ then $V_{\gamma+3}^{\gamma+2}(L)$ would be 2-dimensional.

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1. Introduction

Assume that r, d and g are integers with $0 < 2r < d \leq g - 1$. On a smooth projective algebraic curve C with genus g , the dimension of the scheme C_d^r , can't exceed $d - r$. Through Martens, Mumford and Keem theorems it is known that; proximity of $\dim C_d^r$ to $d - r$, for some r, d ; imposes specific geometry on C . Based on Keem theorem, in the occurrence of $\dim C_d^r = d - r - 2$ for some r and d ; the curve C would be a three sheeted or a 4-sheeted covering of the projective line.

It is unknown whether if one can derive these kind of geometric information for C in the case that; the real dimension of the schemes of Secant Loci associated to an arbitrary very ample line bundle is close to its maximum value. We studied this problem for curves of genus $g \geq 4$ in [4]. After proving Martens theorem for secant loci associated to very ample line bundles on curves of genus $g \geq 4$, we established a Mumford type theorem for curves of genus $g \geq 9$. See [4, Theorem 4.6]. We will complete the next step in this direction, in section 3. Namely, we prove the analogue of Keem's theorem for secant loci associated to very ample line bundles on curves of genus $g \geq 9$. See Theorem 3.5.

Marc Coppens went far beyond this theory in [5], by systematizing Martens, Mumford and Keem theorems. Under some restrictions on the genus of C , he proved that the equality $\dim W_{\gamma+3}^1 = 1$ is guaranteed by an equality $\dim W_d^1 = d - 2 - \gamma$ for an integer γ and some integer d with $0 \leq \gamma + 3 \leq d \leq g - 1 - \gamma$. See problem 2.2. In theorem 4.2 we prove Coppens's result for secant loci when the canonical line bundle of C is replaced by an arbitrary very ample line bundle.

Coppens's method is based on a delicate analysis of a specific irreducible component of W_d^1 . In the absence of a suitable residuation process in our full generality situation, Coppens method seems mostly unapplicable for secant loci of arbitrary line bundles. We take a different approach. Our method relies on an inductive approach together with entering another suitably chosen line bundle in to the argument. Probably the most unexpected advantage of our method is to remove the restrictions, imposed by Coppens, on the genus of C when $\gamma \geq 3$.

In theorem 4.3, we report a dimension computation for secant loci when some specific secant loci are empty. The emptiness assumption is hold for the canonical

line bundle of a general curve, so we re-obtain the classical Brill-Noether dimension theorem for C_d^1 's on general curves.

In order to establish theorems 4.2 and 4.3 we essentially need an analogue of Fulton-Harris-Lazarsfeld result on excess dimension of linear series, for secant loci. See [7]. Such an instrument has been produced only recently by M. Aprodu and E. Sernesi in [2].

Since a divisor $D \in C_{\gamma+3}^1$ gives a $g_{\gamma+3}^1$ on C , replacing $\gamma = 0, 1, 2$, Coppens result specializes to Martens, Mumford and Keem theorems, respectively. Through remark 4.4, we notify that theorem 3.5 can not be concluded from Theorem 4.2, so it actually needs an independent proof.

2. NOTATIONS AND BACKGROUNDS

Assume that L is a line bundle on a smooth projective algebraic curve C of genus g and d a positive integer. For a positive integer $k \leq d - 1$ consider the subset $V_d^k(L)$ of C_d , set theoretically defined by

$$V_d^k(L) := \{D \in C_d \mid h^0(L(-D)) \geq h^0(L) - k\}.$$

The subset $V_d^k(L)$ has a natural scheme structure. See [1], [3], [4] for more details on the scheme structure of $V_d^k(L)$ and some of its geometric properties.

The schemes $V_d^k(L)$ immediately generalize the well known Brill-Noether varieties C_d^r . As well as C_d^r 's the scheme of linear series on an algebraic curve, W_d^r 's, are of central objects in the theory of algebraic curves. C. Keem and M. Coppens have determined non-hyper elliptic curves which for them $\dim W_d^r$ attains one less than its maximum value. See [8], [5].

Theorem 2.1 (Coppens-Keem). *Let C be a smooth algebraic curve of genus $g \geq 9$, and suppose that for some integers d and r satisfying $d \leq g + r - 4, r \geq 1$ we have $\dim W_d^r = d - 2r - 2$. Then C admits a g_4^1 .*

In [5] Marc Coppens, systematizing results of Martens, Mumford and Keem; imposed problem 2.2, concerning dimensions of the varieties of linear series on C , see also [9]:

Problem 2.2. Assume that $g(C) \geq 9$ and γ is a non-negative integer with $2\gamma + 4 \leq g$, $\gamma + 3 \leq d \leq g - 1 - \gamma$. Is it true that; $W_{\gamma+3}^1$ would be 1-dimensional provided that $\dim W_d^1 = d - 2 - \gamma$?

Once the question was treated by Martens and Mumford in the cases $\gamma = 0, 1$ and by Keem in the case $\gamma = 2, g(C) \geq 11$; M. Coppens affirmatively answered it for $\gamma = 2, g(C) = 9, 10$; $\gamma = 3, g(C) = 12, 13, 14$ and $\gamma > 3, g(C) \geq (\gamma + 1)(2\gamma + 1)$. Meanwhile the case $\gamma = 3, g(C) \geq 15$ was answered by Martens.

We prove theorem 4.2, where we slightly generalize and extend M. Coppens and Martens results. We call C an exceptional curve if it is 3-gonal, 4-gonal, bi-elliptic or a space septic curve.

3. KEEM THEOREM FOR SECANT LOCI

In this section, we prove Coppens-Keem theorem for secant loci of very ample line bundles on non-hyper elliptic smooth projective algebraic curves of genus $g \geq 9$.

Lemma 3.1. *Let C be a non-hyper elliptic curve of genus $g \geq 9$. Then C is exceptional provided that $\dim V_3^2(L) = 1$, for some very ample line bundle L on C .*

Proof. This is implicitly contained in the proof of [4, Theorem 4.6]. \square

Example 3.2. For a very ample line bundle L on a bi-elliptic curve C and integers r, d with $0 < 2r < d \leq h^0(L) - 2$, we have $\dim V_d^{d-r}(L) = d - r - 1$. To see this let $\epsilon : C \rightarrow E$ be the elliptic double covering. Then moving p on E , the lines $\langle P, Q \rangle$, where $\epsilon^{-1}(p) = \{P, Q\}$, sweep a cone containing $\phi_L(C)$ in $\mathbb{P}(H^0(L))$, the elliptic cone. Let $l_i = \langle P_i + Q_i \rangle$, for $i = 1, \dots, r+1$, be general generating lines of the elliptic cone such that $\langle l_1, \dots, l_{r+1} \rangle = \mathbf{P}^{r+1} \subset \mathbf{P}(H^0(L))$. Then, for $d \leq h^0(L) - 2$ the divisors of type $D = R_1 + \dots + R_{d-2r-2} + \sum_{i=1}^{r+1} P_i + Q_i$, where R_1, \dots, R_{d-2r-2} are general points on C , belong to $V_d^{d-r}(L)$; implying $\dim V_d^{d-r}(L) = d - r - 1$.

Proposition 3.3. *Let C be a non-hyper elliptic curve of genus $g \geq 4$. Assume that $\dim V_4^3(L) = 1$ for some very ample line bundle L on C . Then C admits a g_d^1 with $d \in \{4, 5, 6\}$.*

Proof. If for some $D \in V_4^3(L)$ one had $h^0(D) = 2$ then C is 4-gonal, while if for each $D \in V_4^3(L)$ we had $h^0(D) = 1$ then $h^0(D_1 + D_2)$ would belong $\{2, 3, 4\}$ for D_1 and D_2 in $V_4^3(L)$. Therefore three cases can occur.

If for general D_1 and D_2 in $V_4^3(L)$ we have $h^0(D_1 + D_2) = 2$ then arguing as in the proof of [4, Theorem 4.6], we obtain a map $\phi : C \rightarrow \mathbf{P}^3$ such that

$$(\deg \phi)(\deg \phi(C) - 1) = 8.$$

According to this equality; if $\deg \phi = 1$, which is the case that $\phi(C)$, as well as C , is a space curve of degree 9; then projecting from a point of C into \mathbf{P}^2 we obtain a singular plane curve of degree 8. Such a curve has to admit a g_6^1 .

If we had $\deg \phi = 2$, which is the same as $\phi(C)$ to be a space quintic, then $\phi(C)$ would admit a g_2^1 and therefore C admits a g_4^1 .

Lastly $\deg \phi = 4$ and $\phi(C)$ is a space cubic curve which has to be a rational normal curve. Therefore C is a 4-sheeted covering of a rational normal space curve, which means that C admits a g_4^1 .

Assume that we are in the second case; i.e. for general $D_1, D_2 \in V_4^3(L)$ we have $h^0(D_1 + D_2) = 3$, by which we conclude that $\dim C_8^2 \geq 2$. This by removing a general point $p \in C$ from the divisors in C_8^2 , implies that $\dim C_7^1 \geq 2$. According to the results of [2], we obtain that C_6^1 is non-empty. Therefore C has to admit a g_6^1 .

Finally we assume that for general $D_1, D_2 \in V_4^3(L)$ one has $h^0(D_1 + D_2) = 4$ and we obtain $\dim C_8^3 \geq 2$. As in the previous, we find that C has to admit a g_5^1 . \square

Lemma 3.4. *For $p \in C$ and a very ample line bundle L on C we have*

$$\dim V_d^{d-1}(L) \leq \dim V_d^{d-1}(L(-p)) \leq \dim V_d^{d-1}(L) + 1.$$

In particular; $V_d^{d-1}(L)$ is non-empty provided that $\dim V_d^{d-1}(L(-p)) \geq 1$ for some $p \in C$.

Proof. The first inequality is immediate from $V_d^{d-1}(L) \subseteq V_d^{d-1}(L(-p))$. To establish the second inequality, we use [2, Theorem 4.1(6)] with $k = d - 1$ and we obtain:

$$(3.1) \quad \dim V_d^{d-1}(L) + 2 \geq \dim V_{d+1}^{d-1}(L).$$

Consider moreover that for $p \in C$ and for any $D \in V_d^{d-1}(L(-p))$ one has $D + p \in V_{d+1}^{d-1}(L)$. This proves $\dim V_d^{d-1}(L(-p)) \leq \dim V_{d+1}^{d-1}(L) - 1$. Comparing with (3.1) we obtain the result. \square

Theorem 3.5. *Assume that C is a smooth projective non-hyper elliptic curve of genus g with $g \geq 9$. If for some very ample line bundle L on C there exist integers r, d with $0 < 2r < d \leq h^0(L) + r - 4$ such that $\dim V_d^{d-r}(L) = d - r - 2$, then C admits a g_d^1 with $d \in \{3, 4, 5, 6\}$.*

Proof. We assume that $V_d^{d-r}(L)$ is irreducible. Removing q from the series in $V_d^{d-r}(L)$ we obtain a $(1 + (d - r - 2) - 1)$ -dimensional family of divisors \bar{D} belonging to $V_{d-1}^{d-r}(L)$, so we would have $\dim V_{d-1}^{d-r}(L) \geq d - r - 2$. This together with [4, Theorem 4.2], implies that either $\dim V_{d-1}^{d-r}(L) = d - r - 2$ or $\dim V_{d-1}^{d-r}(L) = d - r - 1$. Theorem 4.6 of [4] forces C to be exceptional in the latter case. Iterating this process we find that either $\dim V_d^{d-1}(L) = d - 3$ or $\dim V_d^{d-1}(L) = d - 2$, for some $d \leq h^0(L) - 3$. The latter case forcing C to be exceptional, we proceed in the first case.

Assume that L is a very ample line bundle with minimum $h^0(L)$ among those very ample line bundles H , for which $V_d^{d-1}(H)$ is of dimension $d - 3$.

Under this minimality assumption on $h^0(L)$, two cases can occur. For a general $p \in C$ the line bundle $L(-p)$ fails to be very ample, where we would have $\dim V_3^2(L) = 1$ forcing C to be exceptional by Lemma 3.1.

The second possibility is that $h^0(L) = d + 3$. This case consists of three subcases. If for general $p \in C$ the line bundle $L(-p)$ fails to be very ample then we are reduced to the previous case.

The subcase $d = h^0(L) - 3 = 3$ implies that $\dim V_3^2(L) = 0$, which together with [4, Theorem 4.2] implies that either $\dim V_4^3(L) = 1$ or $\dim V_4^3(L) = 2$. Using proposition 3.3 the equality $\dim V_4^3(L) = 1$ implies the assertion. While [4, theorem 4.6], forces C to be exceptional in the case $\dim V_4^3(L) = 2$.

As the last case; assume that for general $p \in C$ the line bundle $L(-p)$ is very ample with $d \geq 4$ and $\dim V_d^{d-1}(L) = d - 3$, where $d = h^0(L) - 3$. Having discussed the case $d = 4$ in Proposition 3.3, we assume that $d \geq 5$. If $\dim V_{d-1}^{d-2}(L) = d - 3$, then C would be exceptional. Assuming $\dim V_{d-1}^{d-2}(L) \neq d - 3$ we make a claim

Claim: $\dim V_d^{d-1}(L(-p)) \neq d - 3$.

Having proved the claim; we use lemma 3.4 together with [4, Theorem 4.2] to obtain

$$\dim V_d^{d-1}(L(-p)) = d - 2.$$

This by [4, Theorem 4.6] forces C to be exceptional. To end the proof; we notify that the bi-elliptic case is excluded via Example 3.2. \square

Proof of the Claim: Equivalent to the claim we prove that; if X is an irreducible component of $V_d^{d-1}(L(-p))$ such that $\dim X = \dim V_d^{d-1}(L(-p))$, then a general member of an irreducible component V of $V_d^{d-1}(L)$ fails to be a general member of X . To do this; consider that for each $D \in V$ there exists $p \in C$ such that $(p + C_{d-1}) \cap V_d^{d-1}(L) \neq \emptyset$. Since $\dim V \geq 2$, moving D in V this p has to move in an open subset of C . Otherwise removing p from the divisors in V we obtain $\dim V_{d-1}^{d-2}(L) = d - 3$, which we had assumed won't occur.

If a general divisor $D \in V$ turns to be a general member of X , then for general $p, q \in C$, divisors of type $D - p + q$ belonging to X lie on V , which is absurd by genericity of q and D . This implies that $d - 3 = \dim V_d^{d-1}(L) < \dim V_d^{d-1}(L(-p))$.

Remark 3.6. (a) Lemma 3.1 shows that, unlike the variety of special divisors, the equality $\dim V_3^2(L) = 1$ for some very ample line bundle L on C , doesn't imply 3-gonality of C .

(b) Based on the proof of Theorem 3.5, we know the shape of a general element in a specific irreducible component of $V_d^{d-1}(L)$, when L is a very ample line bundle on a bi-elliptic curve. This obvious generalization from the canonical case to the case of secant loci of very ample line bundles, remains no longer true when C is 3-gonal or 4-gonal. For example; an easy calculation clarifies that the unique g_3^1 on a 3-gonal curve, as well as a g_4^1 on a 4-gonal curve, does not belong to $V_3^2(2K_C)$, $V_4^3(2K_C)$ respectively. Unfortunately, we don't have any knowledge about the shape of a general member of an element of $V_d^{d-1}(L)$ in the case that C is 3-gonal or 4-gonal.

(c) Each class of curves appeared in theorem 3.5, has a member admitting a very ample line bundle L such that $\dim V_d^r(L) = d - r - 2$ for some integers r, d with $0 < 2r < d \leq h^0(L) + r - 4$.

In fact for a 3-gonal curve of genus g with $d \leq g - 2$, we have $\dim C_d^2 = \dim V_d^{d-2}(K) = d - 4$. See [3, page 198]. For a 4-gonal curve we have $\dim C_d^1 = \dim V_d^{d-1}(K) = d - 3$ with $4 \leq d \leq g - 2$.

On a 5-gonal curve with $p \in C$ as a base point of $K(-g_5^1)$, setting $L = K(-p)$, we observe that for general points q_1, q_2, \dots, q_t on C ; divisors of type $D = g_5^1 + q_1 + q_2 + \dots + q_t$, lie on $V_{t+5}^{t+3}(L)$. This implies that $\dim V_{t+5}^{t+3}(L) = t + 1$.

As in the previous case if C is a 6-gonal curve, then divisors of type $D = g_6^1 + q_1 + q_2 + \dots + q_t$, lie on $V_{t+6}^{t+4}(L)$. This implies that $\dim V_{t+6}^{t+4}(L) \geq t + 1$. If $\dim V_{t+6}^{t+4}(L) = t + 1$, then $V_{t+6}^{t+4}(L(-q))$ would be $t + 2$ dimensional for some $q \in C$.

4. AN IMPROVED ARGUMENT

In this section we affirmatively answer problem 2.2, for secant loci of very ample line bundles. At the same time, removing restrictions on the genus of C imposed by M. Coppens in the spacial case $L = K_C$, we extend it considerably.

Having treated the case $\gamma = 0$ in [4, Theorem 4.2], we will assume that $\gamma \geq 1$.

Lemma 4.1. *Assume that H is a very ample sub-line bundle of the very ample line bundle L such that $\dim V_d^{d-1}(L) = \dim V_d^{d-1}(H)$, for some integer d with $d \leq h^0(L) - 1$. If $V_{d-1}^{d-2}(L)$ is non empty and $\dim V_{d-1}^{d-2}(H) = \dim V_d^{d-2}(H) - 1$, then $\dim V_{d-1}^{d-2}(L) = \dim V_{d-1}^{d-2}(H)$.*

Proof. Assume that X is an irreducible component of $V_d^{d-1}(H)$ in common with $V_d^{d-1}(L)$ such that $\dim X = \dim V_d^{d-1}(H)$. If for some $p \in C$ one had $(p + C_{d-1}) \cap X = X$, then we obtain $\dim V_{d-1}^{d-2}(L) = \dim V_d^{d-1}(H)$, which is impossible. Therefore from the equality

$$X = \cup_{p \in C} [(p + C_{d-1}) \cap X]$$

we conclude that for general $q \in C$, the closed subscheme $Y := (q + C_{d-1}) \cap X$ is of codimension 1 in X . Consider now that, using genericity of q , divisors $D \in C_{d-1}$ such that $q + D \in X$, belong to $V_{d-1}^{d-2}(L)$ and $V_{d-1}^{d-2}(H)$. This implies the assertion.

□

Theorem 4.2. *Let C be a non-hyper elliptic curve of genus $g \geq 9$. Assume that for some very ample line bundle L on C and integers d, γ with $d \geq 3, \gamma \geq 1$ such*

that $h^0(L) \geq 2\gamma + 4$, $\gamma + 3 \leq d \leq h^0(L) - 1 - \gamma$; one has $\dim V_d^{d-1}(L) = d - 1 - \gamma$. Then $V_{\gamma+3}^{\gamma+2}(L)$ has to be 2-dimensional.

Proof. We use induction on γ . Assume $\gamma = 1$. If $L(-p)$ fails to be very ample, then $\dim V_3^2(L) = 1$. The equality $\dim V_4^3(L) = 3$ leads to $\dim V_d^{d-1}(L) \geq d - 1$, which is absurd. Therefore $\dim V_4^3(L) = 2$ and we get the result. Assume $d \geq 4$ and let \mathcal{H} be a very ample sub-line bundle of L with minimum $h^0(\mathcal{H})$ among those very ample sub-line bundles Γ of L such that $\dim V_d^{d-1}(\Gamma) = d - 2$.

For general $p \in C$ if the line bundle $\mathcal{H}(-p)$ fails to be very ample, then as in the previous case we obtain $\dim V_4^3(\mathcal{H}) = 2$. Observing $\dim V_d^{d-1}(\mathcal{H}) = d - 2$ we obtain $\dim V_{\beta+3}^{\beta+2}(\mathcal{H}) = \beta + 1$ for $0 \leq \beta \leq d - 3$. Lemma 4.1 applied to \mathcal{H} and L gives $\dim V_4^3(L) = 2$.

If for general $p \in C$, the line bundle $\mathcal{H}(-p)$ turns to be very ample, then $h^0(\mathcal{H}) = d + 2$ and $\dim V_d^{d-1}(\mathcal{H}) = \dim V_d^{d-1}(\mathcal{H}(-p))$. But an argument as in the proof of our claim in theorem 3.5, excludes this possibility. This completes the case $\gamma = 1$.

Assume $\gamma \geq 2$. We prove that there exists a very ample sub-line bundle Γ of L such that $\dim V_{\gamma+3}^{\gamma+2}(\Gamma) = 2$.

Let \mathcal{H} be choosen as in the case $\gamma = 1$ with $\dim V_d^{d-1}(\mathcal{H}) = d - 1 - \gamma$. If for general $p \in C$ the line bundle $\mathcal{H}(-p)$ fails to be very ample, then we obtain $\dim V_3^2(\mathcal{H}) = 1$. Implying

$$\dim V_d^{d-1}(\mathcal{H}) \geq d - 2 > d - 1 - \gamma = \dim V_d^{d-1}(\mathcal{H}),$$

this case would be absurd. Meanwhile; the possibility $h^0(\mathcal{H}) = d + 1 + \gamma$ with failing very ampleness of $\mathcal{H}(-p)$ for general $p \in C$, is excluded similarly.

Assume that for general $p \in C$ the line bundle $\mathcal{H}(-p)$ is very ample with $\dim V_d^{d-1}(\mathcal{H}) = d - 1 - \gamma$ and $d = h^0(\mathcal{H}) - 1 - \gamma \geq 4$. We distinguish two main cases.

Assuming $\dim V_{d-1}^{d-2}(\mathcal{H}(-p)) = d - 1 - \gamma$ as the first main case, the induction hypothesis implies that $\dim V_{\gamma+2}^{\gamma+1}(\mathcal{H}(-p)) = 2$. Applying [2, Theorem 4.1], $V_{\gamma+3}^{\gamma+2}(\mathcal{H}(-p))$ would be of dimension 3 or 4. The latter case implies $\dim V_d^{d-1}(\mathcal{H}(-p)) \geq 4 + (d - \gamma - 3) = d - \gamma + 1$ which using $\dim V_d^{d-1}(\mathcal{H}) = d - 1 - \gamma$ contradicts lemma 3.4. Therefore $\dim V_{\gamma+3}^{\gamma+2}(\mathcal{H}(-p)) = 3$. Using lemma 3.4, $V_{\gamma+3}^{\gamma+2}(\mathcal{H})$ would be 2 or 3 dimensional. Three dimensionality of $V_{\gamma+3}^{\gamma+2}(\mathcal{H})$ implies that $\dim V_d^{d-1}(\mathcal{H}) \geq d - \gamma$, which is absurd. Therefore $\dim V_{\gamma+3}^{\gamma+2}(\mathcal{H}) = 2$.

As the second main case we assume $\dim V_{d-1}^{d-2}(\mathcal{H}(-p)) \neq d - 1 - \gamma$ and we claim;

$$\dim V_d^{d-1}(\mathcal{H}(-p)) \geq d - \gamma.$$

The claim can be proved as in the proof of our claim in theorem 3.5, so we omit its proof. We use Lemma 3.4 and obtain $\dim V_d^{d-1}(\mathcal{H}(-p)) = d - 1 - (\gamma - 1)$. This again by induction hypothesis asserts; $\dim V_{\gamma+2}^{\gamma+1}(\mathcal{H}(-p)) = 2$, by which as in the first main case we obtain $\dim V_{\gamma+3}^{\gamma+2}(\mathcal{H}) = 2$. To end the proof, we apply Lemma 4.1 and obtain:

$$\dim V_{\gamma+3}^{\gamma+2}(L) = \dim V_{\gamma+3}^{\gamma+2}(\mathcal{H}) = 2.$$

□

Through theorem 4.3, we give an application to our results, specifically theorem 4.2 and lemma 3.4.

Theorem 4.3. *Assume that C is a non-hyper elliptic curve of genus $g \geq 9$ and L is a very ample line bundle on C such that $V_{h^0(L)-d+1}^{h^0(L)-d}(L) = \emptyset$ for some integer d with $\lceil \frac{h^0(L)+3}{2} \rceil \leq d \leq h^0(L) - 1$. Then $\dim V_d^{d-1}(L) = 2d - h^0(L) - 1$.*

In particular; if C is a general curve of genus g , then for integers $d \in \{\lceil \frac{g+3}{2} \rceil, \dots, g-1\}$, the varieties C_d^1 and W_d^1 are of expected dimensions $2d - g - 1$, $2d - g - 2$, respectively.

Proof. We set $d = h^0(L) - k$ and use induction on k . Consider that the case $k = 1$ is immediate by lemma [1, Lemma 2.1]. By our emptiness assumption we obtain that $V_3^2(L) = \emptyset$. Therefore for general $p \in C$, the line bundle $L(-p)$ is very ample. In addition, $V_{h^0(L)-d}^{h^0(L)-d-1}(L(-p))$ contained in $V_{h^0(L)-d+1}^{h^0(L)-d}(L)$ would be empty. Using the induction hypothesis we obtain $\dim V_d^{d-1}(L(-p)) = 2d - h^0(L)$. On use of emptiness of $V_{h^0(L)-d+1}^{h^0(L)-d}(L)$, we find that if X is an irreducible component of $V_d^{d-1}(L(-p))$ such that $\dim X = \dim V_d^{d-1}(L(-p))$, then a general member of an irreducible component, V , of $V_d^{d-1}(L)$ fails to be a general member of X . This, by Lemma 3.4 would give the result.

A general curve of genus g has gonality equal to $\lceil \frac{g+3}{2} \rceil$. Therefore the emptiness assumption is immediate for $L = K_C$ on general curves. \square

Remark 4.4. (a) Although in the case $\gamma = 1$ of theorem 4.2 using [4, Theorem 4.6] we find C an exceptional curve but; based on remark 3.6(b), this is useless to conclude theorem 4.2 when $\gamma = 1$. So we were forced to make extensions in proof of theorem 3.5 to obtain the result directly in the case $\gamma = 1$.

(b) Notice that theorem 3.5 is not a consequence of theorem 4.2 for specific values of γ , e.g. for $\gamma = 1$ or $\gamma = 2$. In fact; in the case $\gamma = 1$, based on theorem 4.2, the equality $\dim V_d^{d-1}(L) = d - 2$ leads to $\dim V_4^3(L) = 2$. This by [2, Theorem 4.1] implies that either $\dim V_3^2(L) = 0$ or $\dim V_3^2(L) = 1$. Although lemma 3.1 implies exceptionality of C in the latter case, but observing the uncontrollable nature of divisors in $V_d^{d-r}(L)$, one can not conclude 3-gonality, 4-gonality or bi-ellipticity of C in the first case. The case $\gamma = 2$ is obviously more complicated.

(c) G. Farkas gives in [6], numerical conditions that they ensure emptiness of $V_{h^0(L)-d+1}^{h^0(L)-d}(L)$ for some integers d and various line bundles on general curves. Therefore theorem 4.3 would be applicable for line bundles and integers having these conditions. Meanwhile there are cases that theorem 4.3 can be applied without using Farkas' results. For example for very ample line bundles on non-exceptional curves of genus $g \geq 9$ we find $\dim V_{h^0(L)-2}^{h^0(L)-3}(L) = d - 3$. Additionally for very ample line bundles on general curves of genus $g \geq 11$, according to the gonality of general curves and using proposition 3.3 we find $\dim V_{h^0(L)-3}^{h^0(L)-4}(L) = d - 4$.

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